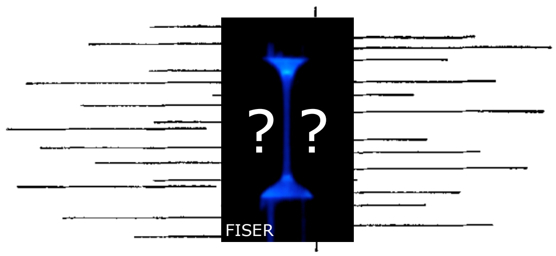
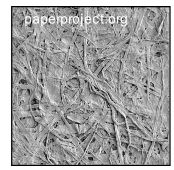
The stress generated in a non-dilute suspension of elongated particles by pure straining motion

by G. K. Batchelor





C. J. Pipe

Hatsopoulos Microfluids Laboratory, MIT

G. K. Batchelor

WIKIPEDIA

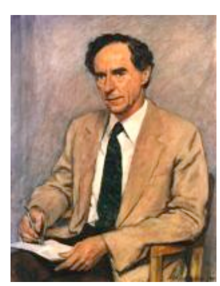
"**George Keith Batchelor (March 8, 1920 - March 30, 2000) was an Australian** applied mathematician and fluid dynamicist. He was for many years the Professor of Applied Mathematics in the University of Cambridge, and was **founding head of the Department of Applied Mathematics and Theoretical Physics (DAMTP)**. **In 1956 he founded the influential Journal of Fluid Mechanics which he edited for some forty years.**

As an applied mathematician (and for some years at Cambridge a co-worker with Sir Geoffrey Taylor in the field of turbulent flow), **he was a keen advocate of the need for physical understanding and sound experimental basis**.

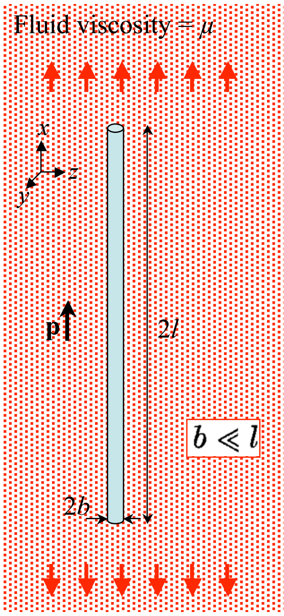
His An Introduction to Fluid Dynamics (CUP, 1967) is still considered a classic of the subject..."

- 1940's: research on turbulence under G. I. Taylor; small scale temperature fluctuations in turbulence, on the dispersion of smoke plumes in a turbulent atmosphere.
- 1960's: research on the statistical distribution of small particles and bubbles as they settle and disperse in liquids and gases - related to problems in chemical engineering and rain in clouds.

From obituary in *The Independent* by J. Hunt



Stress in a solution of elongated particles



Bulk stress due to presence of rod-like particles:

$$\Sigma_{ij}^{(p)} = \frac{1}{V}\Sigma\int_{A_0}\sigma_{ik}x_j n_k dA$$

*traction: $\sigma_{ik}n_k$ is the force per unit area

When the bulk motion causes all particles to align, we can write the approximate relation:

$$\Sigma_{ij}^{(p)} = -\frac{p_i p_j}{V}\Sigma l^2\int_{-1}^{1}\mathbf{p}.\mathbf{F}\, s\, ds$$

Thus the bulk stress in the fluid is:

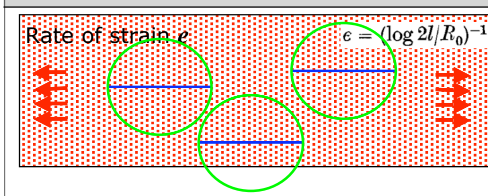
$$\Sigma_{ij} = -P\delta_{ij} + 2\mu e_{ij} + (\tfrac{1}{3}\delta_{ij} - p_i p_j)\frac{1}{V}\Sigma l^2\int_{-1}^{1}\mathbf{p}.\mathbf{F}\, s\, ds$$

Bulk pressure **Viscous stress** **Deviatoric part of particle stress**

Specific viscosity of rods in bulk uniaxial extension flow

$$\frac{\text{Normal stress difference due to particles}}{\text{Viscous normal stress difference}} = \frac{\Sigma_{11}^{(p)} - \frac{1}{2}(\Sigma_{22}^{(p)} + \Sigma_{33}^{(p)})}{3\mu e_{11}}$$

Stress in a dilute suspension



- Particles not affected by each other ('dilute')
- Particles align with extensional flow
- Neglect Brownian motion (Pe→∞/De→∞)
- Steady state flow

$$\mathbf{p}.\mathbf{F} = -\mu p_k p_l e_{kl} l G(s), \longrightarrow \Sigma_{ij}^{(p)} = \frac{\mu e_{kl}}{V}\Sigma p_i p_j p_k p_l l^3\int_{-1}^{1}G(s)\, s\, ds$$

$G(s)$ is non-dimensional function of distance along particle and is found approximately from slender body theory in Stokes flow (Batchelor, 1970b).

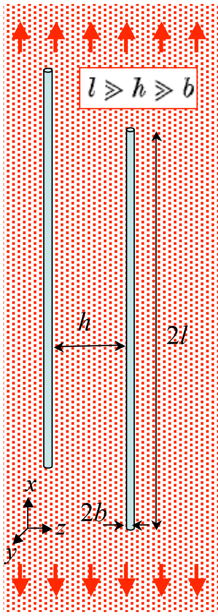
$$\Sigma_{ij}^{(p)} = \delta_{i1}\delta_{j1}\mu e_{11}\alpha \quad \text{where } \alpha = \frac{1}{V}\Sigma\tfrac{4}{3}\pi l^3\epsilon Q(\epsilon) \text{ is the 'volume' fraction of the particles}$$

- Stress due to particles significant for $\alpha \sim 1$
- **Effective particle volume ~ l^3** (and not the actual volume of the particle!) This is the volume that a rod can sweep out around its centre.
- $\alpha \sim 1$ possible, i.e. larger stress than for Newtonian fluid, **but not within the limits of a dilute suspension** - c.f. Einstein's formula for spheres

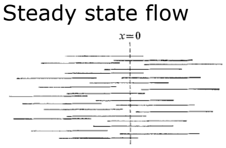
  i.e. for $\alpha \ll 1$ particle stress is small compared with bulk viscous stress.

Experiments show dilute solutions until around 30 times this estimate (Mori *et al* 1982)

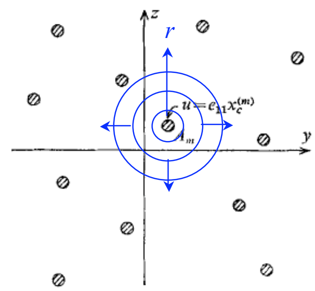
Semi-dilute suspension (close particles)



- **Distance between particles, $h \ll l$**
- Particles align with extensional flow
- Neglect Brownian motion
- Steady state flow



x-y plane



z-y plane

- Spatial velocity gradients important in z-y plane
- Statistically homogeneous
- All fluid points are distance $r<l$ from a rod
- Disturbance is radially symmetric
- Velocity is zero on radius $r = h$

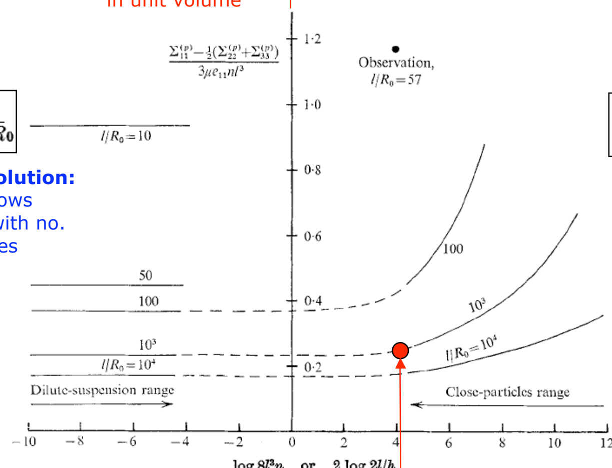
→ **'Cell' model of the flow**

For semi-dilute solution force per unit length: $\Sigma_{ij}^{(p)} = \delta_{j1}\delta_{j1}\mu e_{11}\frac{4\pi}{3V}\sum\frac{l^3}{\log h/R_0}$

cf. result for dilute solution which has $\log(2l/R)$ in denominator

Stress due to particles in extensional flow

Stress per particle in unit volume



$\frac{4\pi Q(\epsilon)}{9\log 2l/R_0}$

Dilute solution: stress grows linearly with no. of particles

$\frac{4\pi/9}{\log 2l/R_0 - \frac{1}{2}\log 8nl^3}$

Close particles: weak departure from stress growing linearly with no. of particles

Quantity of particles

$$\frac{\Sigma_{11}^{(p)}-\frac{1}{2}(\Sigma_{22}^{(p)}+\Sigma_{33}^{(p)})}{3\mu e_{11}} = \frac{4}{9}c\,\frac{l^2/R_0^2}{\log \pi/c}.$$

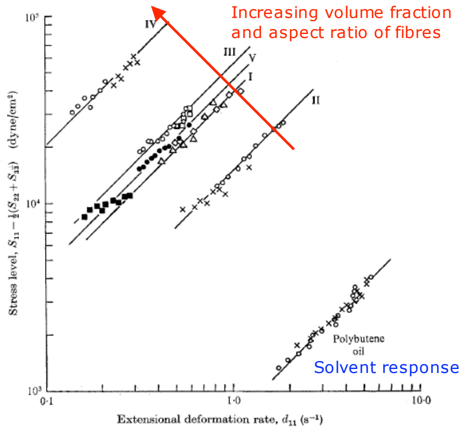
This ratio exceeds 700 when $c = 0{\cdot}01$ and $l/R_0 = 10^3$.
i.e. Trouton ratio = 2100, for 1% volume solution in steady state uniaxial extension

Comparison with experimental data

Rigid rods:
Mewis and Metzner 1974 JFM

Data are presented for the extensional flow of suspensions containing 0·1–1·0% of fibres by volume. The aspect ratio of the fibres was varied from 280 to 1260. The observed stress levels were between one and two orders of magnitude greater than in shearing flow, in agreement with the *a priori* predictions of Batchelor (1971).

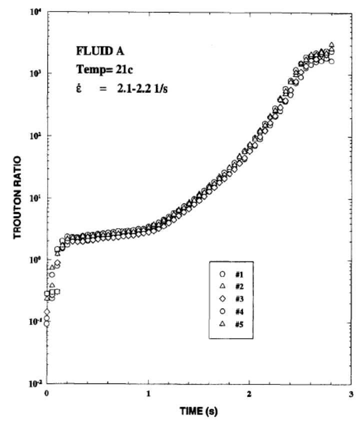
Suspensions of fibres subjected to extensional deformations



Flexible polmers:
James and Sridar 1995 J. Rheol.

TABLE II. Calculations of steady-state T_r values using Batchelor's formula.

Fluid	l μm	ϕ	T_r Batchelor	T_r measured
A	4.3	0.001 14	14 200	~ 2000
B	8.6	0.000 72	34 000	~ 1500
M1	13.6	0.000 93	110 000	



Further results on behaviour of spheroids

- Batchelor's result superceded by results of Shaqfeh and Fredrickson (1990) - improved screening analysis

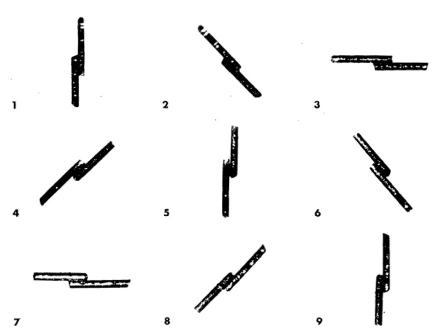
- For *Brownian* solution of spheroidal particles in extension, viscosity decreases by 1.25 (Brenner, 1974)

- Spheroidal particles in shear flow roatate - Jeffery Orbits (Jeffery, 1922), period:

$$f^{-1} = T = \frac{2\pi}{\gamma}\left(r_e + \frac{1}{r_e}\right)$$

- Normal stress differences in steady shearing flow are small (Hinch and Leal, 1971)

- Various results for shear viscosity (Petrie, 1999) due to dilute particles; $\eta/\eta_s \sim 1 + 2\phi$ + H.O.T.



I. Y. Z. Zia, R. G. Cox and S. G. Mason
Proc. Roy. Soc. A. **300**, 421–441 (1967)

Stress due to a particle

Derive particle stress from rate of dissipation.
The rate at which forces at the boundary A_1 do work is:

$$\int_{A_1} e_{ik}\, x_k\, \sigma_{ij}\, n_j\, dA$$

$$e_{ik} \int_{A_1} (-P\delta_{ij} + 2\mu^*\, e_{ij})\, x_k\, n_j\, dA =$$

Equivalent homogeneous bulk fluid

$$e_{ik} \int_{A_1} (-P\delta_{ij} + 2\mu\, e_{ij} + \sigma'_{ij})\, x_k\, n_j\, dA$$

Suspending fluid plus stress due to particle σ'_{ij}

$$2\mu^*\, e_{ij} e_{ij}\, V = 2\mu\, e_{ij} e_{ij}\, V + e_{ij} \int_{A_1} \sigma'_{ij} x_k n_j dA$$

Re-writing for the stress:

$$2\mu^*\, e_{ij} = 2\mu\, e_{ij} + \frac{1}{V} \int_{A_1} \sigma'_{ij}\, x_k\, n_j\, dA$$

This derivation from pp. 251-252 “An Introduction to Fluid Dynamics” Batchelor (1969).
Also see Landau and Lifchitz (1959); even more detailed in Batchelor (1970a).

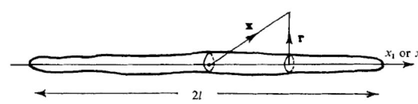
Slender-bodied particles

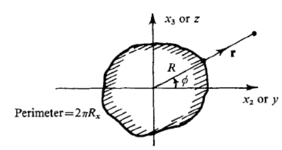
Slender-body theory for particles of arbitrary cross-section in Stokes flow

By G. K. BATCHELOR

J. Fluid Mech. (1970), *vol.* 44, *part* 3, *pp.* 419–440

Slender body theory for Stokes flow: disturbance is approximately that due to a line distribution of Stokeslets.





Flow due to a Stokeslet of force F at the origin:

$$v_j(\mathbf{x}) = \frac{F_j}{8\pi\mu}\left(\frac{\delta_{ij}}{|\mathbf{x}|} + \frac{x_i x_j}{|\mathbf{x}|^3}\right)$$

NB. Disturbance velocity equivalent to Stokes flow at large distances due to a sphere.

$$v_i(\mathbf{x}) = \frac{1}{8\pi\mu}\int_{-l}^{l}\left[\frac{F_i(x')}{\{(x-x')^2+r^2\}^{\frac{1}{2}}} + \frac{(x_i - x_i')(x_j - x_j')\,F_j(x')}{\{(x-x')^2+r^2\}^{\frac{3}{2}}}\right] dx'$$

$$I \approx 2\log\frac{2l}{R_0} + 2\log\frac{(1-x^2/l^2)^{\frac{1}{2}}}{r/R_0}$$ ← approximate solution for integral

Force on a small slender body falling across principal axis is 2 times force when falling along principal axis.

$$F_1 = 2\pi\epsilon\mu U_1, \quad F_i = 4\pi\epsilon\mu U_i \quad (i = 2 \text{ or } 3)$$

$$\epsilon = (\log 2l/R_0)^{-1} \qquad \text{c.f.} \qquad G(s) = \frac{2\pi s}{\log(l/b)} \qquad \text{i.e. } F \sim \varepsilon \sim G(s)$$