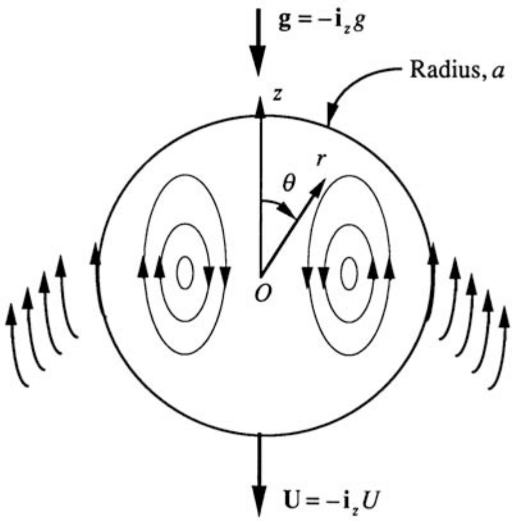
Creeping flow past a liquid sphere



Hadamard (1911)
Comptes rendus de l' académie des sciences V152, p1735

Rybczynski (1911)
Bull. Int. Acad. Pol. Sci. Lett., Cl. Sci. Math. Nat., Ser. A.

Non-Newtonian Fluids Summer Seminar 2007

Philipp Erni
Hatsopoulos Microfluids Laboratory
MIT

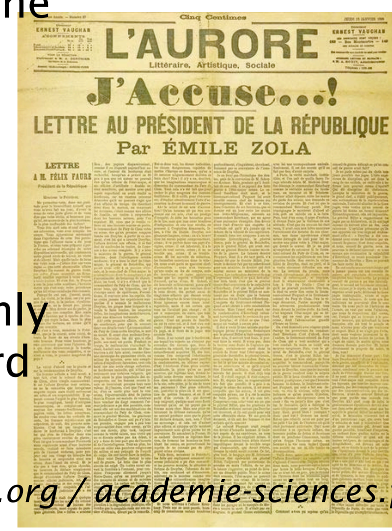
Image from: Edwards, Brenner, Wasan.
'Interfacial transport processes and rheology'

Jacques Salomon Hadamard (1865 – 1963)

French mathematician,
École Normale Supérieure, Paris
Best known for proof of *prime number theorem*
Member of French Academy of Sciences

Other achievements:
- *Hadamard transform* (aka Walsh transform), Hadamard matrix (aka Hadamard gate, used in quantum computers)
- Introduced idea of *well-posed problems* in the theory of PDE
- *Book on Creativity*: “The Mathematician's Mind: The Psychology of Invention in the Mathematical Field” (Dover, 1954)
-*Calculus of variations*

-Brother-in-law of A. Dreyfus (Dreyfus-Affair, a highly divisive political and legal scandal of the French 3rd republic)

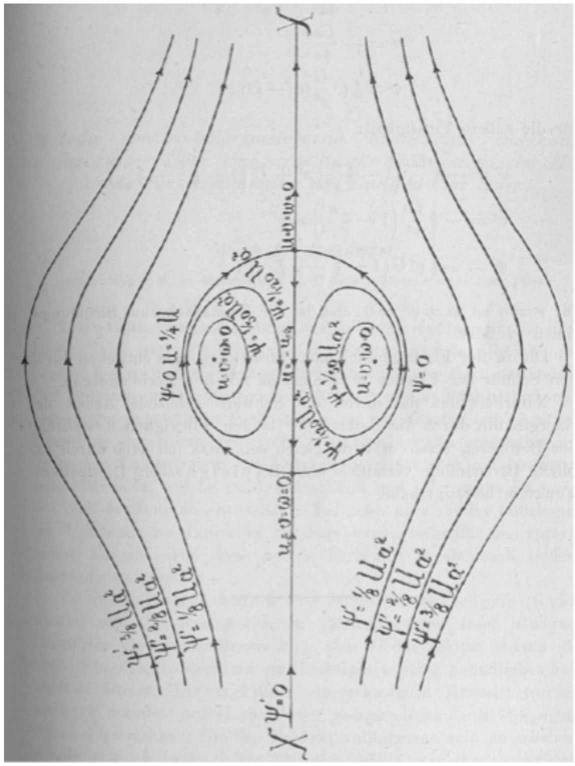


L'AURORE
Littéraire, Artistique, Sociale
J'Accuse...!
LETTRE AU PRÉSIDENT DE LA RÉPUBLIQUE
Par ÉMILE ZOLA

wikipedia.org / academie-sciences.fr

Witolt Rybczynski

-Rybczynski's paper on drops in creeping motion appeared in 1911 in the international bulletin of the Polish Academy of Sciences.
-The article was communicated to the academy by Smoluchowski



O ruchu postępowym kuli ciekłej w ośrodku lepkim. — Über die fortschreitende Bewegung einer flüssigen Kuge[l] in einem zähen Medium.

Note

de M. **WITOLD RYBCZYŃSKI,**

présentée par M. M. Smoluchowski m. c. dans la séance du 9 Janvier 1911.

Das Problem, das wir im folgenden behandeln werden [1]), kan[n] man als eine Verallgemeinerung der bekannten, von Stokes gelöste[n] Aufgabe ansehen, indem wir die Stokes'sche Annahme beibehal[-] ten, daß die Bewegung der Kugel so langsam sei, daß man de[n] Einfluß der Trägheit im Vergleich zu dem Einfluß der Reibun[g] vernachlässigen kann, aber voraussetzen, daß die Kugel kein star[-] rer Körper ist, sondern aus einer reibenden Flüssigkeit besteht.

Da die Lage des Angriffspunktes der wirkenden Kräfte bei de[r] flüssigen Kugel nicht gleichgültig ist, so nehmen wir als Bewe[-]

BULLETIN INTERNATIONAL
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CLASSE DES SCIENCES MATHÉMATIQUES ET NATURELLES
SÉRIE A: SCIENCES MATHÉMATIQUES

ANZEIGER
DER
AKADEMIE DER WISSENSCHAFTEN
IN KRAKAU
MATHEMATISCH-NATURWISSENSCHAFTLICHE KLASSE
REIHE A: MATHEMATISCHE WISSENSCHAFTEN

ANNÉE 1911

CRACOVIE
IMPRIMERIE DE L'UNIVERSITÉ
1911

Problem statement

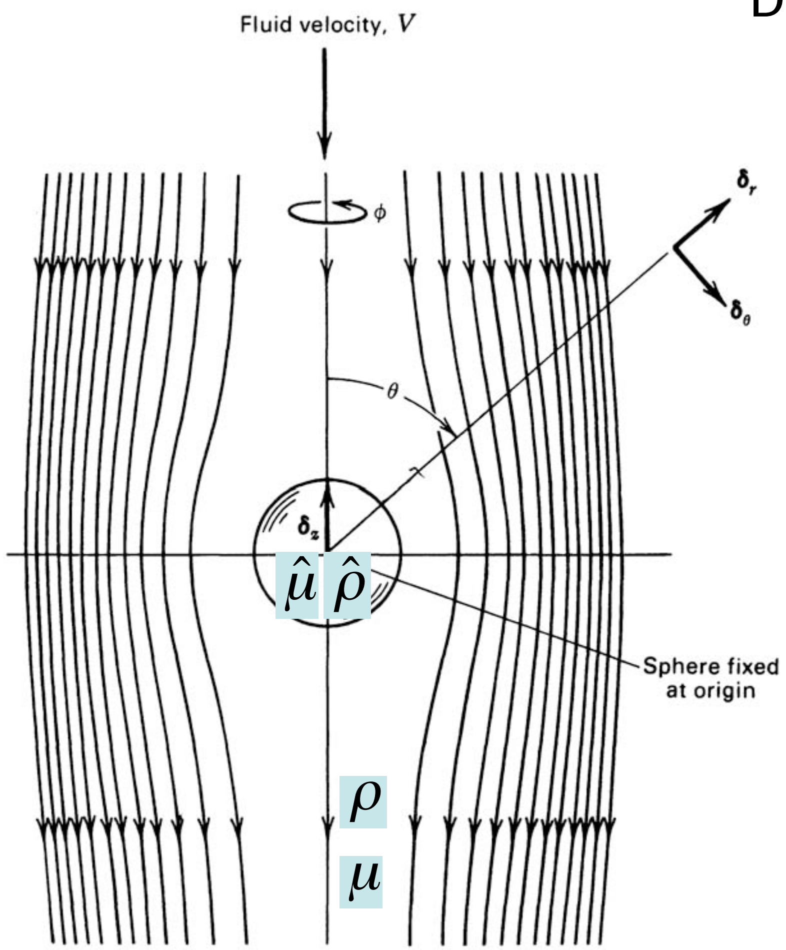


Image: DPL

Drop translating in a second immiscible liquid (under the action of gravity / body force)

Spherical coordinate system (r, θ, φ) with origin in the center of the drop

Axisymmetry: problem in 2D (r,θ) plane

Steady creeping flow: Re << 1

Modification of Stokes' creeping motion problem for a solid sphere to liquid spheres

Stokes (1851):
Drag & settling velocity of a solid sphere

$$\vec{F} = -6\pi r \eta \vec{v} \qquad V_s = \frac{2}{9}\frac{r^2 g(\rho_p - \rho_f)}{\eta}$$

Hadamard/Rybczynski approach

2D Creeping flow: use two stream functions, one for each phase: $\psi, \hat{\psi}$

$$E^4\psi = 0$$

$$E^4\hat{\psi} = 0$$

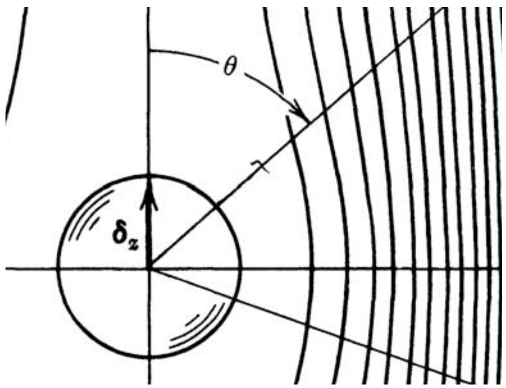
Biharmonic eq.

With operator $E^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$ *("∇^2 in spherical coordinates")*

Velocities in terms of ψ:

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \qquad \hat{v}_r = \frac{1}{r^2\sin\theta}\frac{\partial\hat{\psi}}{\partial\theta}$$

$$v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \qquad \hat{v}_\theta = -\frac{1}{r\sin\theta}\frac{\partial\hat{\psi}}{\partial r}$$



(see tables in *Dynamics of Polymeric Liquids*, p25)

Hadamard/Rybczynski approach

Velocities far from the sphere: $$v_r = -V\cos\theta \qquad v_\theta = V\sin\theta \qquad r \to \infty$$

Use stream functions of the form $\psi = f(r)\sin^2\theta$

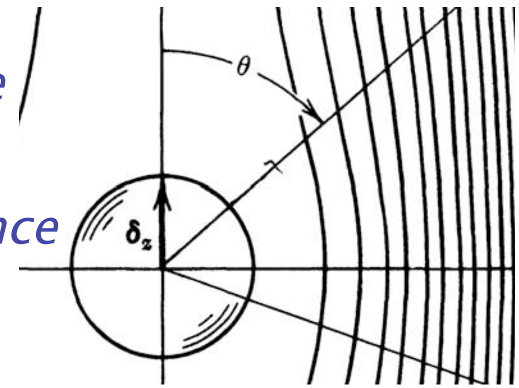
With $f(r) = \left(\frac{A}{r} + Br + Cr^2 + Dr^4\right)$

B.C. at r = a

$$v_r = \hat{v}_r \quad v_\theta = \hat{v}_\theta$$ *"adhesion"*

$$-p + 2\mu\frac{\partial v_r}{\partial r} = -\hat{p} + 2\hat{\mu}\frac{\partial \hat{v}_r}{\partial r}$$ *normal stress balance*

$$\mu\left(r\frac{\partial v_\theta}{\partial r} - v_\theta + \frac{\partial v_r}{\partial \theta}\right) = \hat{\mu}\left(r\frac{\partial \hat{v}_\theta}{\partial r} - \hat{v}_\theta + \frac{\partial \hat{v}_r}{\partial \theta}\right)$$ *tangential stress balance*



(see *DPL*, p27)

Boundary conditions

Use B.C. far away and on drop to eliminate 4 of the 8 constants:
Stream functions inside and outside become

$$\psi = \sin^2\theta\left(\frac{A}{r} + Br\right) \quad \hat{\psi} = \sin^2\theta\left(\hat{C}r^2 + \hat{D}r^4\right)$$

Plug these into the B.C. at the interface:

$$v_r = \hat{v}_r \qquad v_\theta = \hat{v}_\theta \qquad \mu\left(r\frac{\partial v_\theta}{\partial r} - v_\theta + \frac{\partial v_r}{\partial \theta}\right) = \hat{\mu}\left(r\frac{\partial \hat{v}_\theta}{\partial r} - \hat{v}_\theta + \frac{\partial \hat{v}_r}{\partial \theta}\right)$$

(I, II: continuous velocities at r=a)

(III: tangential stresses at r=a)

$$\frac{A}{a} + Ba = \hat{C}a^2 + \hat{D}a^4 \qquad \frac{A}{a^2} - B = -2(\hat{C}a + 2\hat{D}a^3) \qquad \frac{6\mu A}{a^3} = 6\hat{\mu}\hat{D}a^2$$

Boundary conditions

$$-p+2\mu\frac{\partial v_r}{\partial r}=-\hat{p}+2\hat{\mu}\frac{\partial \hat{v}_r}{\partial r}$$ *IV: Normal stress balance at the interface (r=a)*

Pressure from eqs. of motion
$"\nabla^2\mathbf{u}=\nabla p"$

$$\frac{\mu}{\rho}\frac{1}{r\sin\theta}\left[r\frac{\partial(E^2\psi)}{\partial r}d\theta-\frac{1}{r}\frac{\partial(E^2\psi)}{\partial\theta}dr\right]=d(gz-\frac{p}{\rho})$$

$$\frac{\hat{\mu}}{\hat{\rho}}\frac{1}{r\sin\theta}\left[r\frac{\partial(E^2\hat{\psi})}{\partial r}d\theta-\frac{1}{r}\frac{\partial(E^2\hat{\psi})}{\partial\theta}dr\right]=d(gz-\frac{\hat{p}}{\hat{\rho}})$$

with z=rcosθ:

$$p=\left(\rho g a+\frac{2\mu B}{a^2}\right)\cos\theta+const.$$

$$\hat{p}=\left(\hat{\rho} g a+\frac{2\hat{\mu} B}{a^2}\right)\cos\theta+const.$$

$$(\hat{\rho}-\rho)ga=-12\hat{D}\hat{\mu}a+\mu(\frac{12A}{a^4}+\frac{6B}{a^2})$$

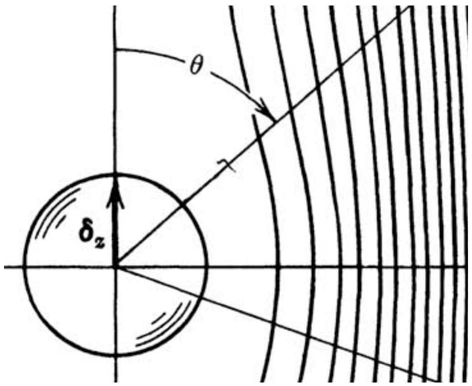
Translational motion (velocity V) + additional motion M, where M is tangential on the surface r=a

$$V=2(\hat{C}+\hat{D}a^2)$$

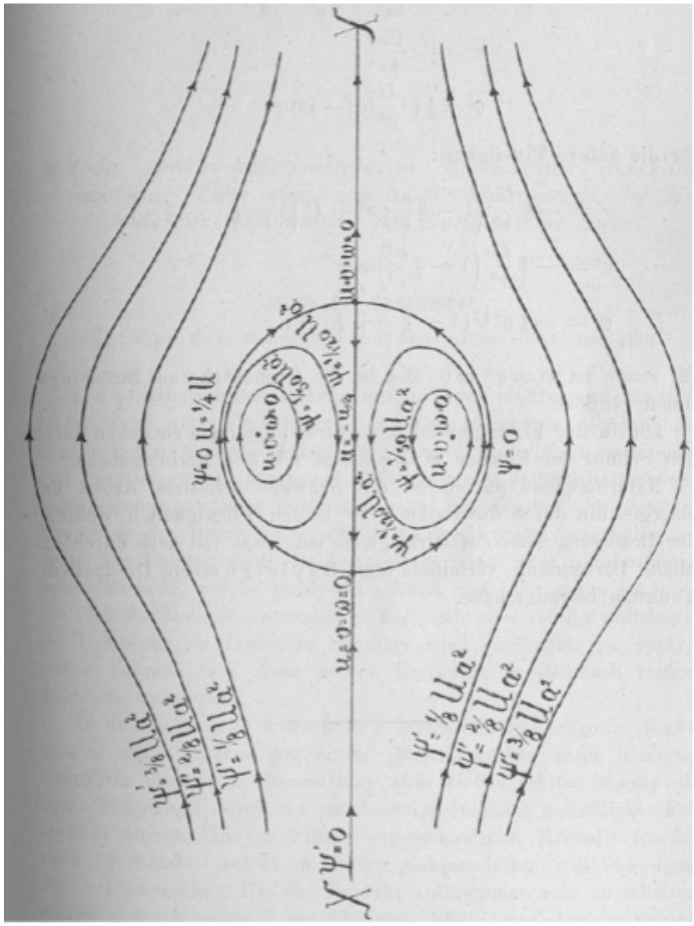
from:

$$\hat{\psi}=\sin^2\theta\left(\hat{C}r^2+\hat{D}r^4\right)$$

$$\hat{v}_r=\frac{1}{r^2\sin\theta}\frac{\partial\hat{\psi}}{\partial\theta}\quad \hat{v}_\theta=-\frac{1}{r\sin\theta}\frac{\partial\hat{\psi}}{\partial r}$$



Translation velocity & streamlines



After some algebra: $$(\hat{\rho}-\rho)g = \frac{9}{2}\mu\frac{V}{a^2}\left(\frac{\mu_1+\frac{2}{3}\mu}{\mu_1+\mu}\right)$$

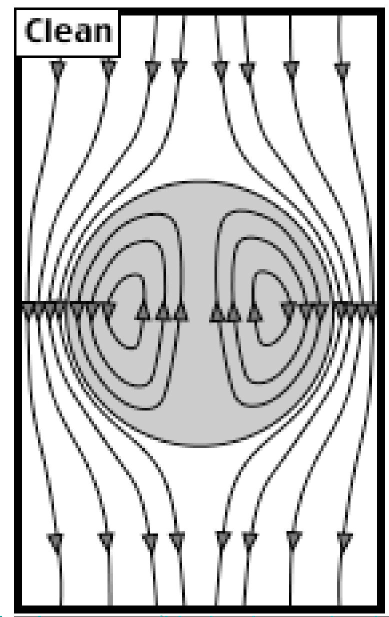
$$V = \frac{2}{9}\frac{\Delta\rho g a^2}{\mu}\left(\frac{\mu_1+\mu}{\mu_1+\frac{2}{3}\mu}\right)$$

(Stokes: $V = \frac{2}{9}\frac{\Delta\rho g a^2}{\mu}$)

Streamlines:

$$r^2(a^2-r^2)\sin^2\theta = const$$

(revolve around a circle in the equator plane with radius $\frac{r}{\sqrt{2}}$



http://www.bubbleology.com/Hydrodynamics.html

Axisymmetric creeping flows in spherical coordinates

For any problem with the stream function ψ in the general form

$$\psi = \sum_{n=1}^{\infty} [A_n r^{n+3} + B_n r^{n+1} + C_n r^{2-n} + D_n r^{-n}] Q_n$$

the force exerted by the fluid on an arbitrary, axisymmetric body with its center of mass at |**x**|=0 is generally given by

$$F_z = 4\pi\mu V_c l_c C_1$$

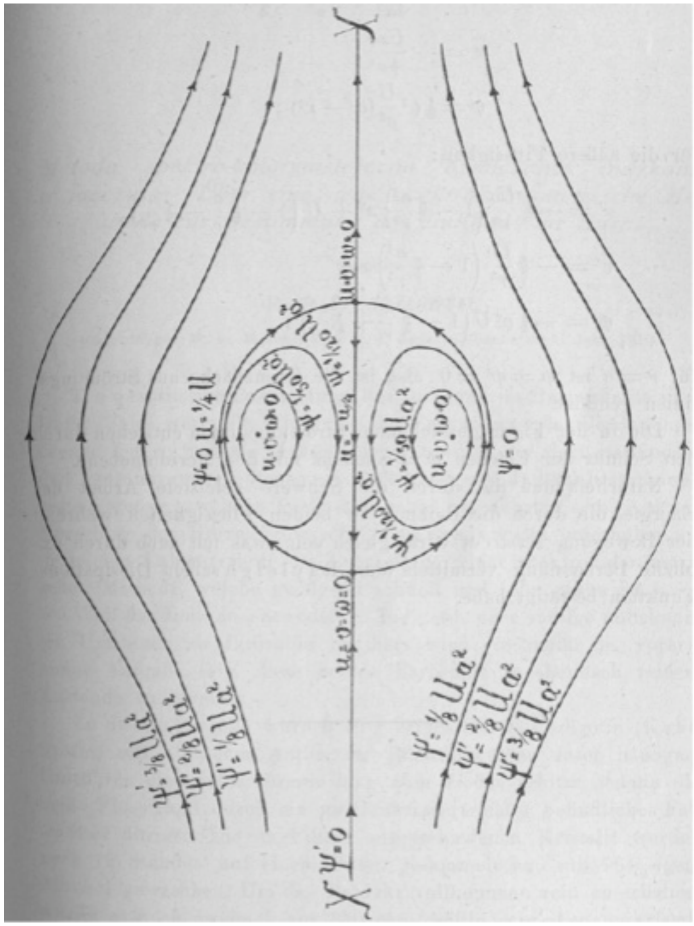
where V_c and l_c are the characteristic velocity length scales and z is the direction of the symmetry axis.

Here: $C_1 = \frac{3\lambda + 2}{2(\lambda + 1)}$ $Drag = F_z = 4\pi\mu Va \frac{3\lambda + 2}{2(\lambda + 1)}$

λ = viscosity ratio innter/outer liquid

LG Leal, *'Laminar Flow and Convective Transport Processes'*, p164 & 209
Deen, *Analysis of Transport Phenomena*

Drag on a drop / Viscosity of Dilute Emulsions



$$Drag = D = 4\pi\mu V a \frac{3\lambda + 2}{2(\lambda + 1)}$$

$$\lambda \to \infty \quad \Rightarrow \quad D = 6\pi\mu a V$$

$$\lambda \to 0 \quad \Rightarrow \quad D = 4\pi\mu a V$$

! Drag on a solid sphere exceeds drag on a spherical bubble by only a factor of 3/2!

Viscosity of a dilute emulsion

$$\eta_r = \frac{\eta}{\eta_s} = 1 + \frac{1 + \frac{5}{2}\lambda}{1 + \lambda}\phi$$

η_r: relative viscosity
η_s: solvent viscosity
λ = viscosity ratio inner/outer
ϕ = volume fraction

Taylor GI (1932), *Proc R Soc Lond A* 138:41

Some examples for later modification: deformation of drop…

Taylor & Acrivos, JFM 1963
- free surface (not spherical)
- must specify kinematic condition for interface

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On the deformation and drag of a falling viscous drop at low Reynolds number

By T. D. TAYLOR
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AND ANDREAS ACRIVOS
Stanford University, Stanford, California

(Received 18 May 1963 and in revised form 23 September 1963)

The motion at low Reynolds number of a drop in a quiescent unbounded fluid is investigated theoretically by means of a singular-perturbation solution of the axisymmetric equations of motion. Special attention is paid to the deformation of the drop. It is shown that for small values of the Weber number *We* the drop will first deform exactly into an oblate spheroid and then, with a further increase in *We*, into a geometry approaching that of a spherical cap. These results are quite insensitive to the ratio of the viscosities of the two fluid phases. The first-order effect of the deformation on the drag of the drop is also included in the analysis.

… internal circulation …

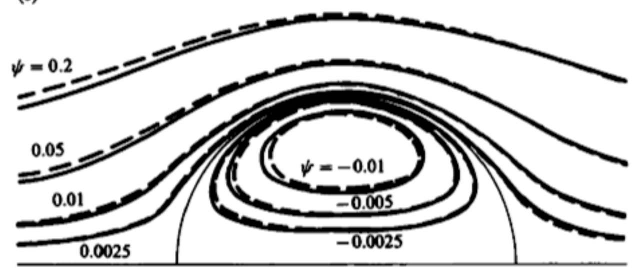
Steady flows inside and around a fluid sphere at low Reynolds numbers

By D. L. R. OLIVER AND J. N. CHUNG
Department of Mechanical Engineering, Washington State University,
Pullman, Washington 99164-2920

(Received 30 November 1983 and in revised form 14 November 1984)

The effects of internal circulation in bubbles and droplets have been analysed by means of a semi-analytical series-truncation method. The equations of motion are transformed into a series of coupled, ordinary, nonlinear differential equations by use of orthogonal sets. These infinite-series equations are then truncated adequately and solved numerically. Using this series-truncation method, we have evaluated the effects of different ratios (between the continuous and dispersed phases) of both density and viscosity for the flows of low Reynolds numbers. For all the density ratios investigated, the density difference has almost no effect on the drag coefficient at low Reynolds numbers. The shear stress and the drag coefficient increase with increasing viscosity ratio of droplet to ambience and decrease with increasing Reynolds number.

(c)



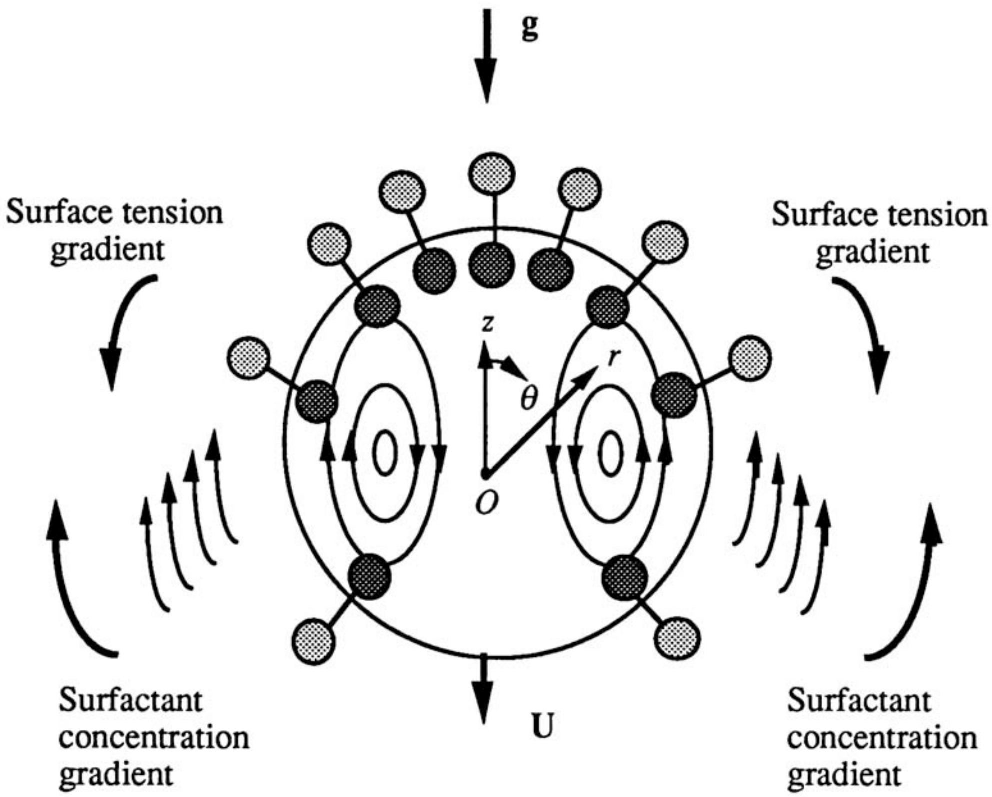
2. Basic Assumptions

(*a*) The droplet remains spherical (Rivkin *et al.* (1976) cite experimental evidence that droplets stay reasonably spherical if the Weber number is below 0.1).
(*b*) Both fluids are Newtonian and mutually immiscible, and there is no chemical reaction.
(*c*) The system involves only purified fluids (there are no surface-active materials).
(*d*) There is no interfacial mass transfer (the radial velocity is zero at the interface).
(*e*) Fluid properties are constant and the flow is steady.

Drag coefficient at low but non-zero Re

$$C_D = C_{Do} + 0.0126(C_{Do}\, Re_o)^2.$$

...Drops with surfactants



Adsorption-controlled
Marangoni flow

$$\mathrm{Re} = \frac{2a\rho V}{\mu} << 1$$

$$Pe_S = \frac{2aV}{D^S} >> 1$$

Surface tension gradients appear
In the B.C.
→need surface equation of state
→e.g. Frumkin, Szyckowski

$$\gamma_0 - \gamma_{eq} = -RT\Gamma_\infty \ln\left(1 - \frac{\Gamma_{eq}}{\Gamma_\infty}\right)$$

Choice of stream function

Why $f(r) = \left(\frac{A}{r} + Br + Cr^2 + Dr^4 \right)$?

$$\psi = f(r)\sin^2\theta \qquad E^4\psi = 0$$

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial\theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \right) \right] \left[\sin^2\theta \left(\frac{d^2 f}{dr^2} - \frac{2f}{r^2} \right) \right] = 0 \quad \Rightarrow \quad \left(\frac{d^2}{dr^2} - \frac{2}{r^2} \right) \left(\frac{d^2}{dr^2} - \frac{2}{r^2} \right) f = 0$$

'equidimensional eq.', has solutions of form $f = r^n$
with [n(n−1)−2][(n−2)(n−3)−2]=0, therefore n = −1,2,3,4

(see chapter 'Drag on a sphere' in *DPL*)